Math Preliminaries (Solutions)

FIZIKA SPhO Training

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1 Notes

1.1 Limits

Mathematically speaking, a limit is concerned with the behaviour of a function as it *approaches* a certain value. What does "approach" mean? Physically, it suffices to say "get arbitrarily close to your value of interest".

The notation is

$$\lim_{x \to a} f(x) = L \tag{1}$$

which means, in English, as x gets arbitrarily close to a, $f(x)$ gets closer and closer to L.

1.1.1 Basic Techniques

We shall begin with a few simple examples.

Example 1.1. Evaluate

$$\lim_{x \to 2} \left(\sin \left(\sqrt{x+4} \right) \right)$$

Simply substitute in $x = 2$ to get $\sin \left(\sqrt{6} \right)$. Nothing fancy at all.

Example 1.2. Evaluate

$$\lim_{x \to 2} \left(\frac{x^3 - 8}{x - 2} \right)$$

Substituting $x = 2$ doesn't really work as you'll get $\frac{0}{0}$. Instead, realise that $x^3 - 8 = (x-2)(x^2 + 2x + 4)$, thus

$$\lim_{x \to 2} \left(\frac{x^3 - 8}{x - 2} \right) = \lim_{x \to 2} \left(x^2 + 2x + 4 \right) = 12$$

Example 1.3. Evaluate

$$\lim_{x \to \infty} \left(\frac{2x^2 + 3x + 1}{5x^2 + 8} \right)$$

Dividing by x^2 in the numerator and denominator, we get

$$\lim_{x \to \infty} \left(\frac{2x^2 + 3x + 1}{5x^2 + 8} \right) = \lim_{x \to \infty} \left(\frac{2 + \frac{3}{x} + \frac{1}{x^2}}{5 + \frac{8}{x^2}} \right)$$

As $x \to \infty$, the terms containing $\frac{1}{x}$ and $\frac{1}{x^2}$ grow negligibly small and approach 0, thus

$$\lim_{x \to \infty} \left(\frac{2 + \frac{3}{x} + \frac{1}{x^2}}{5 + \frac{8}{x^2}} \right) = \frac{2}{5}$$

1.1.2 L'Hopital's Rule

A commonly used technique to solve limits is **L'Hopital's Rule.** It states that when

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right)$$

is of an indeterminate form (either $\frac{0}{0}$ or $\frac{\infty}{\infty}$), then

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \lim_{x \to a} \left(\frac{f'(x)}{g'(x)} \right) \tag{2}$$

provided f and g are differentiable in some interval containing a, but not necessarily at a itself.

Example 1.4. Using L'Hopital's Rule, solve Example 1.2 and 1.3.

For Example 1.2, using L'Hopital's Rule,

$$\lim_{x\to 2}\left(\frac{x^3-8}{x-2}\right)=\lim_{x\to 2}\left(\frac{3x^2}{1}\right)=12$$

For Example 1.3, using L'Hopital's Rule,

$$\lim_{x\to\infty}\left(\frac{2x^2+3x+1}{5x^2+8}\right)=\lim_{x\to\infty}\left(\frac{4x+3}{10x}\right)=\lim_{x\to\infty}\left(\frac{4}{10}\right)=\frac{2}{5}$$

These examples may seem contrived for physics problems, but they are meant to get your mathematical techniques in shape. In physics, the limits we deal with are usually **infinite limits** (to $\pm\infty$), as that gives us the asymptotic behaviour of a system after a long time. To analyse these behaviours, you need to know how to evaluate your limits!

1.1.3 Two Must-Know Limits

Two famous limits you should know for physics are:

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1 \tag{3}$$

$$\lim_{x\to 0}\left(\frac{\cos x-1}{x}\right)=0 \tag{4}$$

Note that L'Hopital's Rule **cannot** be used to prove these limits - it is circular reasoning!

1.1.4 Application: Damped Harmonic Oscillator

When we cover oscillations, you will see that the equation of motion of a damped harmonic oscillator is

$$x(t)=Ae^{-\gamma t}\cos(\omega t+\phi),\ \gamma>0$$

If you are interested in finding its behaviour after a long time, you may take an infinite limit:

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}Ae^{-\gamma t}\cos(\omega t+\phi)=0$$

since $e^{-\gamma t}$ approaches 0 as t approaches ∞. Physically, this makes sense - after a long time, the damping force has fully dissipated the motion, so that it stops moving and settles back to its equilibrium position, $x=0$.

1.1.5 Application: Falling with Drag

Most of the time, we assume drag force to be linear in velocity, so that $F_{drag}=-bv$. It can be shown that the velocity as a function of time for a falling object starting at rest is

$$v(t)=\frac{mg}{b}\left(1-e^{-\frac{b}{m}t}\right)$$

If you are interested in finding the velocity of the object after a long time (known as terminal velocity), you may take an infinite limit:

$$\lim_{t\to\infty}v(t)=\lim_{t\to\infty}\frac{mg}{b}\left(1-e^{-\frac{b}{m}t}\right)=\frac{mg}{b}$$

since $e^{-\frac{b}{m}t}$ approaches 0 as t approaches ∞. Physically, this makes sense - this is the point where $mg=bv$, which means the drag force and gravitational force cancel, giving no acceleration, thus the velocity remains constant.

1.2 Derivatives

A derivative is concerned with the rate of change of a quantity with respect to another.

1.2.1 First Principles

The formal definition of the derivative is given by a limit. It states that

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) \tag{5}$$

or, in some books, equivalently,

$$f'(a) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right) \tag{6}$$

This is called **differentiation from first principles.**

Example 1.5. From first principles, for $f(x) = \sin x$, find $f'(x)$.

Using Equation (5) above,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left(\frac{\sin(x+h)-\sin x}{h}\right) = \lim_{h\to 0}\left(\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{(\sin x)(\cos h - 1)}{h} + (\cos x)\frac{\sin h}{h}\right) = (\sin x)\lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + (\cos x)\lim_{h\to 0}\left(\frac{\sin h}{h}\right) \\ &= \cos x \end{aligned}$$

where the last equality holds using Equations (3) and (4).

1.2.2 Second Principles

What you are probably more familiar with is called **differentiation from second principles.** This includes stuff from math in school, like your product rule, quotient rule, and chain rule:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \tag{7}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - g'(x)f(x)}{(g(x))^2} \tag{8}$$

$$(f\,(g(x)))' = f'\,(g(x))\,g'(x) \tag{9}$$

And a few common derivatives that you should memorise by heart:

$\frac{d}{dx}n = 0$	$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}x = 1$	$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}x^n = nx^{n-1}$	$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$
$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}\cot x = -\csc^2 x$	$\frac{d}{dx}\mathrm{arccot}\, x = -\frac{1}{1+x^2}$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\mathrm{arcsec}\, x = \frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}n^x = n^x \ln x$	$\frac{d}{dx}\csc x = -\csc x \cot x$	$\frac{d}{dx}\mathrm{arccsc}\, x = -\frac{1}{x\sqrt{x^2-1}}$

Revisit your H2 math if needed.

1.2.3 Implicit Differentiation

Sometimes, in physics, we may not be able to find a neat, explicit expression for y in terms of x (some quantity in terms of another). We may turn to implicit differentiation in this case.

Essentially, the idea is to treat y as a function of x and use the chain rule.

Example 1.6. Find $\frac{dy}{dx}$ given that

$$2x\cos(y) + 3\sin(xy) = 5$$

Using implicit differentiation, differentiate both sides with respect to x, then

$$2\cos(y) - 2x\sin(y)\frac{dy}{dx} + 3\cos(xy)\left(y + x\frac{dy}{dx}\right) = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2\cos(y) + 3y\cos(xy)}{2x\sin(y) - 3x\cos(xy)}$$

provided that $2x\sin(y) \neq 3x\cos(xy)$.

1.2.4 Application: Force and Potential Energy

In later chapters, you will see that for **conservative** forces in 1D, the force F and potential energy U are related by

$$F = -\frac{dU}{dx}$$

That means, given any contrived potential energy function, you can find the associated force!

For instance, if

$$U(x) = 2\sin\left(x^3\right) + 4x\tan\left(\frac{1}{x}\right)$$

then the associated force is

$$F(x) = -\frac{d}{dx}\left(2\sin\left(x^3\right) + 4x\tan\left(\frac{1}{x}\right)\right) = -6x^2\cos\left(x^3\right) - 4\tan\left(\frac{1}{x}\right) + \frac{4}{x}\sec^2\left(\frac{1}{x}\right)$$

1.3 Maclaurin and Taylor Series

These are very important in physics as they are used for **approximations.** Hereby, we shall use $f^n(x)$ to denote the n-th derivative of $f(x)$.

1.3.1 Maclaurin Series

A **Maclaurin series** is always centered around $x = 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + ... = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n \tag{10}$$

1.3.2 Taylor Series

While a **Taylor series** is a "generalised Maclaurin series", centered around $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \tag{11}$$

In physics, you'll never need the infinite form of the series. The main point about making an approximation is to reduce any complex function into a polynomial, which is easier to work with.

1.3.3 Common Approximations

The most common approximations you should know are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... \tag{12}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... \tag{13}$$

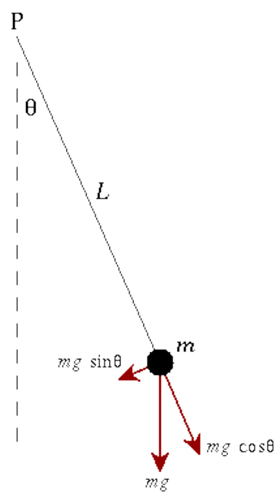
$$(1+x)^n = 1 + nx + \frac{1}{2}n(n-1)x^2 + ... \quad \text{where } |x| < 1 \tag{14}$$

Equation (14) is especially important! You'll see its first-order expansion a lot in physics. A common clue to invoke it is if the problem states that $|x| << 1$ ($|x|$ much smaller than 1).

Remark. Sometimes, you'll need to consider terms higher than the first non-zero order. Unfortunately, it may be difficult to determine which order you should Taylor expand to. One way to find out is to realise that your result makes no physical sense if you haven't expanded to a sufficient order. You should then revisit the step where you expanded, and go to higher order.

1.3.4 Application: Small Angle Pendulum

Consider a small pendulum bob of mass m, on a string of length L, displaced by an angle of $\theta << 1$ from the vertical.



It can be shown that, using Newton's 2nd Law, the equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

However, you'll see when we cover ODEs that this isn't very easy to solve. Instead, since we are given that $\theta << 1$, we may use the Maclaurin series for $\sin\theta$, to approximate $\sin\theta \approx \theta$ to first-order. This is accurate enough since θ is small enough!

Thus, we may rewrite the above equation as

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

which you will realise later on that it is much easier to solve.

1.4 Integrals

Integration is the reverse procedure to differentiation. We'll focus on refining your integration techniques in this section.

1.4.1 Common Integrals

You should memorise all of these by heart.

Common Integrals

$\int k\,dx = kx + c$	$\int \cos u\,du = \sin u + c$	$\int \tan u\,du = \ln\lvert\sec u\rvert + c$
$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$	$\int \sin u\,du = -\cos u + c$	$\int \sec u\,du = \ln\lvert\sec u + \tan u\rvert + c$
$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$	$\int \sec^2 u\,du = \tan u + c$	$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$
$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\lvert ax+b\rvert + c$	$\int \sec u \tan u\,du = \sec u + c$	$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + c$
$\int \ln u\,du = u\ln(u) - u + c$	$\int \csc u \cot u\,du = -\csc u + c$	
$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$	$\int \csc^2 u\,du = -\cot u + c$	

1.4.2 Integration by Substitution

Commonly referred to as u-substitution, this method relies on a clever substitution to eliminate parts of your integral, reducing it to something more elementary.

The rule of thumb is to usually let u be something in the denominator, or some strange expression hidden in some function. Any non-trivial substitutions (or even non-trivial integrals) will usually be given to you for Physics Olympiad.

Example 1.7. Find

$$\int \frac{x^2+1}{x^3+3x}\,dx$$

Let $u = x^3 + 3x$. Then, $\frac{du}{dx} = 3x^2 + 3 = 3\left(x^2+1\right)$. Thus,

$$\int \frac{x^2+1}{x^3+3x}\,dx = \int \frac{x^2+1}{u}\,\frac{du}{3\left(x^2+1\right)} = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln\lvert u\rvert + C = \frac{1}{3}\ln\lvert x^3+3x\rvert + C$$

Don't forget to convert back to x!

Example 1.8. Evaluate

$$\int_0^9 \sqrt{4-\sqrt{x}}\,dx$$

Let $u = 4 - \sqrt{x}$. Then, $\frac{du}{dx} = -\frac{1}{2\sqrt{x}} = -\frac{1}{2(4-u)}$. Additionally, $4-\sqrt{0} = 4$, $4 - \sqrt{9} = 1$. Thus,

$$\int_0^9 \sqrt{4-\sqrt{x}}\,dx = -2\int_4^1 \sqrt{u}\,(4-u)\,du = 2\int_1^4 \left(4u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)\,du = \ldots = \frac{188}{15}$$

Don't forget to change the bounds for a definite integral!

1.4.3 Trigonometric Substitution

You should learn to spot some common forms of integrands that call for trigonometric substitution. This means substituting x to be some trigonometric function (for example, $\sin\theta$).

Be familiar with your trigonometric identities. In particular, the Pythagorean ones will be used:

$$\sin^2 x + \cos^2 x = 1 \tag{15}$$

$$1 + \tan^2 x = \sec^2 x \tag{16}$$

$$1 + \cot^2 x = \csc^2 x \tag{17}$$

Given a to be some constant, the common forms for trigonometric substitution are:

$$a^2 - x^2 \quad \text{or} \quad \sqrt{a^2 - x^2} \quad \longrightarrow \quad \text{sub } x = a\sin\theta \tag{18}$$

$$a^2 + x^2 \quad \text{or} \quad \sqrt{a^2 + x^2} \quad \longrightarrow \quad \text{sub } x = a\tan\theta \tag{19}$$

$$x^2 - a^2 \quad \text{or} \quad \sqrt{x^2 - a^2} \quad \longrightarrow \quad \text{sub } x = a\sec\theta \tag{20}$$

As usual, you'll find dx in terms of $d\theta$ and replace all x with θ. Using Equations (15) to (17), you can simplify the integrand, and then solve.

Example 1.9. Find

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\, dx$$

By first completing the square, we see that

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\, dx = \int \frac{1}{\sqrt{9(x-2)^2 + 1}}\, dx$$

Let $x - 2 = \frac{1}{3}\tan\theta$. Then, $\frac{dx}{d\theta} = \frac{1}{3}\sec^2\theta$. Thus,

$$\int \frac{1}{\sqrt{9(x-2)^2 + 1}}\, dx = \frac{1}{3}\int \frac{1}{\sqrt{\tan^2\theta + 1}}\, \sec^2\theta\, d\theta = \frac{1}{3}\int \sec\theta\, d\theta$$

$$= \frac{1}{3}\ln|\sec\theta + \tan\theta| + C = \frac{1}{3}\ln\left|\sqrt{9(x-2)^2 + 1} + 3(x-2)\right| + C$$

1.4.4 Integration by Parts (IBP)

When all else fails, you might need to resort to IBP. Given an integrand with a product of functions, IBP reduces it as such:

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx \tag{21}$$

Usually, you'll choose $f(x)$ to be the function that is easy to differentiate (and possibly hard to integrate), and $g'(x)$ to be the function that is easy to integrate.

One such guideline that exists is **LIATE.**

L	Logarithmic	(like $\log(x)$)
I	Inverse trigonometric	(like $\arctan(x)$)
A	Algebraic	(like $5x^2 + 3$)
T	Trigonometric	(like $\cos(x)$)
E	Exponential	(like 10^x)

This gives the order in which you should choose $f(x)$ (i.e. the function to be differentiated). Logarithmic is most preferable, and exponential is least preferable.

Of course, this is just a guideline. There may be exceptions to this!

Example 1.10. Find

$$\int e^{3x} \sin(2x)\, dx$$

Using IBP, let $f(x) = \sin(2x)$ and $g'(x) = e^{3x}$. Then, $f'(x) = 2\cos(2x)$ and $g(x) = \frac{1}{3}e^{3x}$. Thus,

$$\int e^{3x} \sin(2x)\, dx = \frac{1}{3}e^{3x}\sin(2x) - \frac{2}{3}\int e^{3x}\cos(2x)\, dx$$

Now, let $f(x) = \cos(2x)$ and $g'(x) = e^{3x}$. Then, $f'(x) = -2\sin(2x)$ and $g(x) = \frac{1}{3}e^{3x}$. Thus,

$$\int e^{3x} \cos(2x)\, dx = \frac{1}{3}e^{3x}\cos(2x) + \frac{2}{3}\int e^{3x}\sin(2x)\, dx$$

Now, we realise that if we substitute this back into the first equation, we obtain

$$\int e^{3x} \sin(2x)\, dx = \frac{1}{3}e^{3x}\sin(2x) - \frac{2}{3}\left(\frac{1}{3}e^{3x}\cos(2x) + \frac{2}{3}\int e^{3x}\sin(2x)\, dx\right)$$

which is a linear equation in the desired integral. Solving the equation, we obtain

$$\int e^{3x} \sin(2x)\, dx = \frac{1}{13}e^{3x}\left(3\sin(2x) - 2\cos(2x)\right) + C$$

1.4.5 Application: Work Done

You'll see in later chapters that the work done by a **conservative** force in 1D is given by

$$W = \int F\, dx$$

As an example, consider a force $F(x) = x\cos(x)$. The work done by this force from $x = 0$ to $x = \pi$ is

$$W = \int_0^{\pi} x\cos(x)\, dx = -2$$

Verifying this is left as an exercise for you.

1.5 Ordinary Differential Equations (ODEs)

ODEs pop up way too commonly in physics. Most things are not linear (like in algebra-based physics). The rate of change of a quantity may depend on the value of some other quantity, or the quantity itself, making it difficult to solve.

You will need to know how to solve **first and second-order linear differential equations.** Conventionally, primed notation (y') is a derivative with respect to position, while dot notation ($\dot{y}$) is a derivative with respect to time.

1.5.1 First-order ODEs

A first-order ODE is of the form

$$\frac{dy}{dx} + yP(x) = Q(x) \tag{22}$$

where $P(x)$ and $Q(x)$ are some functions of x. Let's investigate some cases.

Case 1: $P(x)$ is a constant. Let $P(x) = a$. Consider first the *homogeneous* differential equation (just set the right hand side to be 0),

$$\frac{dy}{dx} + ay = 0 \tag{23}$$

By separating variables,

$$\int \frac{dy}{y} = \int -a\,dx \quad \Rightarrow \quad y_{homogeneous} = Ke^{-ax} \text{ for some constant } K \tag{24}$$

Now, we consider the *particular* solution. You want to guess a particular solution depending on what $Q(x)$ is. Here are some examples, where A, B, C are constants to be determined.

Q(x)	**Particular Solution**
2	A
$5x+7$	$Ax+B$
$5x^2+7$	Ax^2+Bx+C
$5\cos(2x+3)$	$A\cos(2x)+B\sin(2x)$
$5e^{2x+3}$	Ae^{2x}

After guessing your particular solution, plug the form of your guess into Equation (22), and solve for the unknown constants. By doing so, you find $y_{particular}$.

Conclude by adding your homogeneous and particular solutions, to find

$$y = y_{homogeneous} + y_{particular} \tag{25}$$

and use the initial conditions provided to find any remaining constants.

Case 2: $P(x)$ is not a constant. Here, we will use the method of **integrating factor.** The integrating factor is defined as

$$I(x) = e^{\int P(x)\,dx} \tag{26}$$

It can be shown that, by multiplying $I(x)$ to both sides, Equation (22) can be reduced to

$$\frac{d}{dx}\left(yI(x)\right) = Q(x)I(x) \tag{27}$$

which can be solved by separation of variables by moving dx to the right.

You may realise that Case 1 could have been solved using an integrating factor. But, through this, we hope that you are exposed to the homogeneous + particular solution method too.

Example 1.11. Solve the ODE

$$3xy^2\frac{dy}{dx} + 8x^2 = \ln x$$

One might notice that we don't need any of the fancy tricks above! Notice that

$$3xy^2\frac{dy}{dx}+8x^2=\ln x \quad\Rightarrow\quad y^2\,dy=\frac{\ln x-8x^2}{3x}\,dx=\left(\frac{1}{3}\frac{\ln x}{x}-\frac{8}{3}x\right)\,dx$$

We can just integrate both sides to find $y(x)$. The rest of the steps are left as an exercise for you. You may leave your answer in terms of constants since no initial conditions were specified.

You should eventually obtain

$$y=\left(\frac{1}{2}(\ln x)^2-4x^2+C\right)^{\frac{1}{3}}$$

Example 1.12. Solve the ODE

$$x\frac{dy}{dx}-3y=x^5$$

Dividing both sides by x, and then using an integrating factor,

$$I(x)=e^{\int -\frac{3}{x}\,dx}=e^{-3\ln|x|}=\frac{1}{x^3}$$

Thus, multiplying both sides by $I(x)$, the equation becomes

$$\frac{1}{x^3}\frac{dy}{dx}-\frac{3y}{x^4}=x \quad\Rightarrow\quad \frac{d}{dx}\left(\frac{y}{x^3}\right)=x \quad\Rightarrow\quad \frac{y}{x^3}=\frac{x^2}{2}+C \quad\Rightarrow\quad y=\frac{x^5}{2}+Cx^3$$

1.5.2 Second-order ODEs

A second-order ODE is of the form

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+yQ(x)=R(x) \tag{28}$$

where $P(x)$, $Q(x)$ and $R(x)$ are some functions of x. Let's investigate some cases.

For Physics Olympiad, we'll only deal with the case where $P(x)$, $Q(x)$ and $R(x)$ are constants. This is solved using **Euler's Method,** where we guess an exponential solution. Let $P(x)=a$, $Q(x)=b$ and $R(x)=c$.

First, let's consider the *homogeneous* differential equation (replace the right-hand-side by 0). Guess $y=Ae^{sx}$. Then, the equation reduces to

$$Ae^{sx}\left(s^2+as+b\right)=0 \quad\Rightarrow\quad s^2+as+b=0 \tag{29}$$

Solving this quadratic equation (also known as the **characteristic equation**) for s will yield 2 solutions, s_1 and s_2. There are 3 possible cases.

Case 1: s_1 and s_2 are unique, real roots. The general homogeneous solution is

$$y_{homogeneous}=Ae^{s_1x}+Be^{s_2x} \tag{30}$$

where A and B are constants to be determined.

Case 2: s_1 and s_2 are repeated, real roots. Then, $s=s_1=s_2$. The general homogeneous solution is

$$y_{homogeneous}=Ae^{sx}+Bxe^{sx} \tag{31}$$

where A and B are constants to be determined.

Case 3: s_1 and s_2 are unique, complex roots. Suppose s (either s_1 or s_2) is $c \pm di$. The general homogeneous solution is

$$y_{homogeneous} = e^{cx}\left(A\cos(dx) + B\sin(dx)\right) \tag{32}$$

where A and B are constants to be determined.

To find the particular solution, guess one based on the same principles in Section 1.5.1, and determine the unknown coefficients.

Finally, add the homogeneous and particular solutions, then substitute in initial conditions to obtain the final solution for y.

Example 1.13. Solve the ODE

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 14y = 18e^{2x}$$

Using the guess $y = Ae^{sx}$, the characteristic equation is

$$s^2 + 5s - 14 = 0$$

Solving this quadratic gives $s = 2, -7$. Thus,

$$y_{homogeneous} = Ae^{2x} + Be^{-7x}$$

For the particular solution, this is quite subtle. You would probably first guess $y_{particular} = Ce^{2x}$. Then, substituting this back in,

$$4Ce^{2x} + 5\left(2Ce^{2x}\right) - 14Ce^{2x} = 18e^{2x} \quad \Rightarrow \quad 0 = 18e^{2x}\,???$$

Clearly, this makes no sense. This means your guess was wrong! In this case, you'll make another guess, $y_{particular} = Cxe^{2x}$. Substituting this in,

$$\left(4C + 4Cx\right)e^{2x} + 5\left(C + 2Cx\right)e^{2x} - 14\left(Cx\right)e^{2x} = 18e^{2x} \quad \Rightarrow \quad 9C = 18 \quad \Rightarrow \quad C = 2$$

Thus, the particular solution is actually $y_{particular} = 2xe^{2x}$. The general solution is

$$y = y_{homogeneous} + y_{particular} = Ae^{2x} + Be^{-7x} + 2xe^{2x}$$

This example teaches you an important lesson about guessing particular solutions. If your guess leads to an equation that makes no sense, you need to multiply by x (or even x^2 and above, if all else fails). The explanation is quite mathematically involved; it is called resonance. You don't need to know the reasoning, you just need to remember this exists.

1.5.3 Application: RC Circuits

An RC circuit consists of a resistor and a capacitor. We'll go through this in later handouts, but it can be shown that for a capacitor C with initial charge Q_0 discharging through a resistor R, the differential equation is

$$\frac{dQ}{dt} + \frac{Q}{RC} = 0$$

Solving this first-order ODE with the initial condition $Q(0) = Q_0$ gives

$$Q(t) = Q_0 e^{-\frac{t}{RC}}$$

1.6 Manipulating Differentials

It's important for you to be comfortable with dealing with infinitesimal quantities and differentials.

1.6.1 $\frac{dy}{dx}$ Is A Fraction

As you have seen above, differentials are treated like normal quantities. This means you are *technically* allowed to treat $\frac{dy}{dx}$ as a fraction!

For example, you know that speed is given by $v = \frac{dx}{dt}$. Treating $\frac{dx}{dt}$ as a fraction means it is legal to write

$$dx = v\,dt$$

and you may integrate both sides. This technique, called separation of variables, is useful if v is a function of t.

Treating it as a fraction also allows you to apply the chain rule. For example, to express the rate of mass changing over time, $\frac{dm}{dt}$, we may write

$$\frac{dm}{dt} = \frac{dm}{dx}\frac{dx}{dt} = v\frac{dm}{dx}$$

which could be more convenient if we already know the rate of mass changing over distance, $\frac{dm}{dx}$.

1.6.2 Applying A Differential

The art of manipulating differentials also means you can do other cool things by just "applying a differential". For instance, consider kinetic energy, $K = \frac{1}{2}mv^2$. Manipulating differentials allows us to write

$$dK = \frac{1}{2}v^2\,dm + mv\,dv$$

by just differentiating each side and applying the product and chain rules.

This idea is very important. All physical laws can technically be written in terms of differentials, to analyse infinitesimal components of systems.

1.6.3 Application: Thermodynamics

Thermodynamic laws will be covered in later handouts. However, it is very common to express them in terms of differentials.

The first law of thermodynamics is commonly expressed as

$$dU = dQ + dW_{on} = T\,dS - p\,dV$$

Expressing in differentials is more convenient as you can easily set up differential equations afterwards.

2 Problems

Problems are arranged in roughly increasing difficulty.

Problem 2.1. Write this equation (known as the Larmor Formula) in terms of differentials, assuming that physical constants remain constant.

$$P = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3}$$

Solution. Simply taking differentials on both sides, we find

$$dP = \frac{1}{6\pi\varepsilon_0 c^3}\left(2qa^2\,dq + 2aq^2\,da\right) = \frac{1}{3\pi\varepsilon_0 c^3}\left(qa^2\,dq + aq^2\,da\right)$$

Problem 2.2 (Ricardo)**.** The electric field along the axis of a disk of radius R, carrying a charge Q uniformly distributed over the disk, is given by

$$E = kQ\frac{x}{(x^2+R^2)^{\frac{3}{2}}}$$

where k is a constant and x is the distance from the center of the disk. By how much will the field decrease at the point x if the radius of the disk is increased by a small amount ΔR, while the charge Q is kept constant?

Solution. Taking differentials, keeping k, Q and x constant, we have

$$dE = kQx\,d\left((x^2+R^2)^{-\frac{3}{2}}\right) = -\frac{3}{2}kQx(x^2+R^2)^{-\frac{5}{2}}(2R\,dR) = -3kQxR(x^2+R^2)^{-\frac{5}{2}}\,dR$$

Thus, the field *decreases* by

$$\Delta E = \frac{3kQxR\Delta R}{(x^2+R^2)^{\frac{5}{2}}}$$

Problem 2.3. Find $\frac{dy}{dx}$ for

$$y = \frac{1}{\sqrt{1-\csc(\ln x)}}$$

Solution. Simply differentiating, applying the quotient rule and chain rule gives

$$\frac{dy}{dx} = \frac{\left(\sqrt{1-\csc(\ln x)}\right)(0) - \frac{d}{dx}\left(\sqrt{1-\csc(\ln x)}\right)(1)}{1-\csc(\ln x)} = -\frac{\frac{1}{2\sqrt{1-\csc(\ln x)}}\frac{d}{dx}\left(1-\csc(\ln x)\right)}{1-\csc(\ln x)}$$

$$= -\frac{\frac{1}{2\sqrt{1-\csc(\ln x)}}\left(\csc(\ln x)\cot(\ln x)\frac{d}{dx}(\ln x)\right)}{1-\csc(\ln x)} = -\frac{\csc(\ln x)\cot(\ln x)}{2x\left(1-\csc(\ln x)\right)^{\frac{3}{2}}}$$

Problem 2.4. Up till the x^5 term, find the Taylor series, centered around $x=-4$, of

$$f(x) = e^{-6x}$$

Solution. Notice that the exponential differentiates nicely:

$$f'(x) = -6e^{-6x},\ f''(x) = 36e^{-6x},\ f'''(x) = -216e^{-6x} \quad \Rightarrow \quad f^n(x) = (-6)^n e^{-6x}$$

Thus, applying the Taylor series,

$$f(x) \approx f(-4) + f'(-4)(x+4) + \frac{f''(-4)}{2!}(x+4)^2 + \frac{f'''(-4)}{3!}(x+4)^3 + \frac{f^4(-4)}{4!}(x+4)^4$$

$$+\frac{f^5(-4)}{5!}(x+4)^5 = e^{24}\left(1 - 6(x+4) + 18(x+4)^2 - 36(x+4)^3 + 54(x+4)^4 - \frac{324}{5}(x+4)^5\right)$$

Problem 2.5. Up till the x^5 term, find the Maclaurin series of

$$f(x) = (1+x)^{-\frac{1}{2}}$$

Solution. Notice that

$$f'(x) = -\frac{1}{2}(1+x)^{-\frac{3}{2}},\ f''(x) = \frac{3}{4}(1+x)^{-\frac{5}{2}},\ f'''(x) = -\frac{15}{8}(1+x)^{-\frac{7}{2}},$$

$$f^4(x) = \frac{105}{16}(1+x)^{-\frac{9}{2}},\ f^5(x) = -\frac{945}{32}(1+x)^{-\frac{11}{2}}$$

Thus, applying the Maclaurin series,

$$\begin{aligned} f(x) &\approx f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5 \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 - \frac{63}{256}x^5 \end{aligned}$$

Problem 2.6. Find $\frac{dy}{dx}$ for

$$\tan\left(x^2y^4\right) = 3x + y^2$$

Solution. Using implicit differentiation, differentiating both sides with respect to x,

$$\sec^2\left(x^2y^4\right)\left(2xy^4 + 4x^2y^3\frac{dy}{dx}\right) = 3 + 2y\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3 - 2xy^4\sec^2\left(x^2y^4\right)}{4x^2y^3\sec^2\left(x^2y^4\right) - 2y}$$

provided that $4x^2y^3\sec^2\left(x^2y^4\right) \neq 2y$.

Problem 2.7. Find

$$\int (\ln x)^2\ dx$$

(Hint: Can you find a substitution? If you can't, what should you try next?)

Solution. Using integration by parts with $f(x) = (\ln x)^2$, $g'(x) = 1$, then

$$f'(x) = \frac{2\ln x}{x},\ g(x) = x$$

Thus,

$$\int (\ln x)^2\ dx = x(\ln x)^2 - 2\int \ln x\, dx = x(\ln x)^2 - 2x\ln x + 2x + C$$

Problem 2.8. Find

$$\int \frac{5\cos^3 x + 2\sin^3 x}{2\sin^2 x\cos^2 x}\, dx$$

(Hint: Manipulate the integrand before proceeding!)

Solution. Splitting the fraction first,

$$\int \frac{5\cos^3 x + 2\sin^3 x}{2\sin^2 x\cos^2 x}\, dx = \int \left(\frac{5\cos x}{2\sin^2 x} + \frac{\sin x}{\cos^2 x}\right) dx$$

Let $u = \sin x$. Then, $\frac{du}{dx} = \cos x$. Thus,

$$\int \frac{5\cos x}{2\sin^2 x}\, dx = \frac{5}{2}\int \frac{\cos x}{u^2}\frac{du}{\cos x} = \frac{5}{2}\int \frac{1}{u^2}\, du = -\frac{5}{2u} + C_1 = -\frac{5}{2\sin x} + C_1 = -\frac{5}{2}\csc x + C_1$$

Let $u = \cos x$. Then, $\frac{du}{dx} = -\sin x$. Thus,

$$\int \frac{\sin x}{\cos^2 x}\, dx = \int \frac{\sin x}{u^2} \frac{du}{-\sin x} = -\int \frac{1}{u^2}\, du = \frac{1}{u} + C_2 = \frac{1}{\cos x} + C_2 = \sec x + C_2$$

In total, we have

$$\int \frac{5\cos^3 x + 2\sin^3 x}{2\sin^2 x \cos^2 x}\, dx = -\frac{5}{2}\csc x + \sec x + C$$

Problem 2.9. Solve the ODE

$$\ddot{x} + 6\dot{x} + 25x = 104e^{3t}$$

Solution. First, consider the homogeneous equation. The characteristic equation is

$$s^2 + 6s + 25 = 0 \quad \Rightarrow \quad s = -3 \pm 4i$$

Thus, the homogeneous solution is

$$x_{homogeneous} = e^{-3t}\left(A\cos(4t) + B\sin(4t)\right)$$

For the particular solution, guess $x_{particular} = Ce^{3t}$. Then,

$$(9 + 18 + 25)Ce^{3t} = 104e^{3t} \quad \Rightarrow \quad C = 2$$

$$x = x_{homogeneous} + x_{particular} = e^{-3t}\left(A\cos(4t) + B\sin(4t)\right) + 2e^{3t}$$

Problem 2.10. Solve the ODE

$$y'' + 5y' - 9y = e^{-2x} + 2 - x$$

Solution. First, consider the homogeneous equation. The characteristic equation is

$$s^2 + 5s - 9 = 0 \quad \Rightarrow \quad s = \frac{-5 \pm \sqrt{61}}{2}$$

Call these roots s_1 and s_2 for convenience. Thus, the homogeneous solution is

$$y_{homogeneous} = Ae^{s_1 x} + Be^{s_2 x}$$

For the particular solution, guess $x_{particular} = Ce^{-2x} + Dx + E$. Then,

$$-15Ce^{-2x} - 9Dx + (5D - 9E) = e^{-2x} + 2 - x \quad \Rightarrow \quad C = -\frac{1}{15},\ D = \frac{1}{9},\ E = -\frac{13}{81}$$

$$y = y_{homogeneous} + y_{particular} = Ae^{s_1 x} + Be^{s_2 x} - \frac{1}{15}e^{-2x} + \frac{1}{9}x - \frac{13}{81}$$

Problem 2.11. Solve the ODE

$$\frac{d^2y}{dx^2} + 9y = 18\sin(3x) + 12\cos(3x)$$

Solution. First, consider the homogeneous equation. The characteristic equation is

$$s^2 + 9 = 0 \quad \Rightarrow \quad s = \pm 3i$$

Thus, the homogeneous solution is

$$y_{homogeneous} = A\cos(3x) + B\sin(3x)$$

For the particular solution, guess $y_{particular} = x\left(C\cos(3x) + D\sin(3x)\right)$. We tried a solution with the x already multiplied, because of resonance. If you are not convinced, try it without the x, and you'll get nonsensical results. So, with this guess,

$$-6C\sin(3x) + 6D\cos(3x) = 18\sin(3x) + 12\cos(3x) \quad \Rightarrow \quad C = -3,\ D = 2$$

$$y = y_{homogeneous} + y_{particular} = A\cos(3x) + B\sin(3x) + x\left(-3\cos(3x) + 2\sin(3x)\right)$$